

The Time-Consistency of Government Debt and Institutional Restrictions on the Level of Debt

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Abstract

The higher is the level of government debt the larger the government's incentives to default or to devalue. Is it desirable to limit the amount of government debt? To answer this question, we focus on the whole set of sustainable equilibria. We find that a limit on debt can be beneficial as it can eliminate some bad equilibria. In particular, it can rule out default. On the other hand, as the worst sustainable equilibrium is now better, a limit on debt makes a deviation from the announced policy less painful and, in turn, may make the Ramsey outcome not sustainable. Applying the APS method, we show that the Ramsey can be sustained with an additional limit on the budget deficit.

Keywords: Debt Limits; Debt Management; Optimal Policy; Rules vs. Discretion; Time-Consistency

JEL classification: E61, E62, H21, H62, H63

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1 Introduction

Since Kydland and Prescott (1977)'s advocacy for rules rather than discretion, there has been a lot of debate about how much we should tie the hands of the policymaker in order to reduce or eliminate time-inconsistency problems.¹ In this paper we study the time-inconsistency problem of government debt. As earlier literature illustrates,² governments are tempted to default on and to devalue (through the manipulation of the interest rate) their debt obligations. Moreover, the higher is the level of government debt the larger the government's temptation. Is then desirable to impose an institutional constraint on the amount of government debt?

Many countries have recently adopted limits on government debt.³ There are two main arguments in favor, namely, (i) these debt limits may lessen time-inconsistency problems; and (ii) they may help prevent excessive government spending.⁴ The main argument against is that they may reduce the flexibility of the government in response to shocks. This paper attempts to isolate the effect of debt limits on the time-inconsistency problem of government debt.

We consider an environment similar to that of Lucas and Stokey (1983), a deterministic economy where a benevolent government must decide how to finance the government spending through taxes on labor income and through the issue of debt. In the economy with commitment, debt limits are generally not desirable. If binding, they limit the ability of the government to smooth taxes over time and, hence, reduce welfare.⁵ In the economy without commitment, the government is tempted to default and to manipulate the interest rate of the outstanding debt obligations. We assume that a default on debt payments entails a direct cost to the economy, more precisely, a productivity fall.⁶ In this

¹The establishment of rules or institutional restrictions reduces the set of policies that the policymakers can set and, in turn, they may choose not to renege on their previous announcements. The seminal papers of Barro and Gordon (1983) and Rogoff (1985) study the effect of rules when there are no mechanisms to commit to such rules. Here we follow Athey, Atkinson and Kehoe (2005) and assume that the rule is legislated and complied by the governments.

²See Prescott (1977) and Lucas and Stokey (1983) for an earlier exposition of these issues.

³For example, in the US, public expenditure (included interest payments) cannot exceed tax revenue. In the European Monetary Union countries, the Stability Pact requires that their government budget deficits and government debt should not be greater than 3% and 60% of GDP, respectively.

⁴See Buiter, Corsetti and Roubini (1993).

⁵This argument can be found in Marcet and Scott (2003) and Stockman (2004).

⁶Sturzenegger (2004) estimates the real effects of default in the 80s and finds that output drops an accumulated 4% over the following 4 years after default.

setup, we assume that the society can credibly impose a limit on government debt and explore whether such institutional restriction is desirable.

To answer this question, we focus on the whole set of sustainable equilibria rather than the best sustainable equilibrium.⁷ We do so because the implementation of one sustainable equilibrium - out of a large set of sustainable equilibria - requires a considerable coordination of beliefs by private individuals and such coordination is not under the control of the government.⁸ We apply the APS method developed by Abreu, Pearce and Stacchetti (1990) and extended to dynamic policy games by Chang (1998), Phelan and Stacchetti (2001) and Sleet (1997) to our economy. We first study the economy without any restrictions on the level of government debt. There we find that the worst sustainable equilibrium is a balanced budget with or without default. This is so because the government can always deviate and unilaterally impose a balanced budget. Next, we consider an institutional restriction on the level of government debt. We find that a debt limit changes the set of sustainable equilibria. In particular, a debt limit can make default not longer sustainable. Moreover, since the worst sustainable equilibrium provides now higher welfare, a debt limit may make some good equilibria, e.g. the Ramsey outcome, not sustainable. Thus, our findings capture what once Rogoff (1987) pointed out, we obtain that debt limits have the “danger of throwing out the baby (any good reputational equilibria) with the bathwater”.⁹

We next explore under which conditions a debt limit can mitigate the time-inconsistency problem of interest rate manipulation. In particular, we allow for an additional restriction on the government budget deficit. Then we find that the worst sustainable equilibrium is characterized by a more irregular path of consumption that yields less welfare than the balanced budget. The punishment after a deviation can be more severe and a debt limit may eliminate some bad equilibria without ruling out some good ones. This is not the only possibility however. The equilibrium set of values depends on the specific form of the budget deficit constraint. Our results show that restrictions to the budget deficit can be used to reduce the set of sustainable equilibria and, in this way, help coordinate the expectations of the individuals.

⁷Krusell, Martín and Ríos-Rull (2005) and Martín (2006) focus on Markov perfect equilibria to study the time-inconsistency problem of interest rate manipulation. Here we allow for default and for non-Markov equilibria.

⁸For a discussion of this point see Chari (1988), Rogoff (1987) and Stokey (1989).

⁹Rogoff (1987) suggests that this may happen when restrictions on the public’s expectations are imposed so as to lessen the multiplicity of supergame equilibria. He also suggests institutional restrictions as a means of reducing this multiplicity.

A lesson to learn from this work is that in the design of institutional arrangements (to alleviate time-inconsistency problems), one should explore which hands are tied, whether we are tying those of the public or of the government, and by how much. The first rule, a debt limit, is indeed an instrument to the government that allows him to tie the expectations of the individuals' away from default and, moreover, achieve a very good outcome after a deviation. The second, a budget deficit, imposes a restriction on the government and affects the expectations of the individuals. If well designed, it can be an effective instrument against the time-inconsistency problem of interest rate manipulation.

Our research builds on earlier studies on the time-consistency of optimal policy. One of the most important essays is that of Lucas and Stokey (1983). They study the time-inconsistency problem of manipulating the interest rate on government debt and show that the careful management of the maturity of debt can make the optimal policy time-consistent. Their work has been continued by several papers. Among them, Chari and Kehoe (1993a) is the first to introduce a reputational mechanism in the Lucas and Stokey's setup to study the problem of default on government debt. Their main result is that a reputation cannot sustain positive debt. In contrast, Chari and Kehoe (1993b) assume that individuals can also default, which implies non-negative government debt. This lower limit on debt changes the set of sustainable equilibria, the worst sustainable equilibrium is autarky, and the need to smooth taxes over time helps sustaining positive debt in equilibrium. The present paper blends the time-inconsistency problems studied in Lucas and Stokey (1983) and Chari and Kehoe (1993a,b). We also have non-negative debt but assume a direct cost of default. Similarly we find that the imposition of an upper limit on debt changes drastically the set of sustainable equilibrium. In our setup, however, a government chooses to deviate through interest rate manipulation rather than default, and no default characterizes the worst sustainable equilibrium.

One of the earliest works on balanced budget constraints and time-consistency is that of Towe (1989). In a non-Ricardian economy, he studies whether a balanced budget constraint can make the discretionary policy time-consistent. He finds that balanced budget rules which impose stationarity on the optimal policy solve the time-inconsistency problem. Our paper extends his analysis by studying the effects of these rules on the whole set of sustainable equilibria.

This paper is closely related to that of Stockman (2004), which studies the problem of default in a stochastic environment. He considers a very tight balanced budget rule, namely, that debt must be constant, which binds with and without commitment. He finds that, since this rule restricts the possibilities of smoothing the cost of distortionary

taxation, it lowers the gains from continuing with the announced policy. Therefore, he concludes that balanced budget rules make most likely a default on government debt in equilibrium. Since this is not observed in the data, he suggests that there may be other mechanisms sustaining a no-default equilibrium. The difference between his results and ours lies in the costs of default. In Stockman's paper, the only cost of defaulting is the inability to smooth taxes over time. Such inability is also present in the balanced budget rule. Therefore, unless the rule is state-contingent, the government always prefers to default. In contrast, this paper considers that a default also brings a productivity fall in the economy. Given this cost, the government may prefer to stick to a balanced budget rather than default. This explains our results and, furthermore, reconciles Stockman's results with the data.

Sleet and Yeltekin (2006) consider the time-inconsistency problems of default by individuals and government and truthful revelation of information by the government. They focus on the best sustainable incentive-compatible allocations and find that they are characterized by endogenous lower and upper limits on government debt. These limits are necessary because incentive-compatibility makes the debt level drift over time. Above the upper limit, the government defaults and below the lower one, the individuals do the same. A key difference between this paper and theirs is the direct cost of default. With this cost, an upper limit on debt changes the set of sustainable equilibria. This introduces a role for a debt limit which is not present in their paper.

Two recent papers that study constitutional constraints and credibility in monetary policy are those of Chari and Kehoe (2007) and Athey, Atkenson and Kehoe (2005). Chari and Kehoe (2007) study the time-inconsistency problem of price level setting in monetary unions with fiscal constraints. They obtain that, if the monetary authority can commit, then a debt limit only brings costs. However, if they cannot commit, then the monetary union produces a free-rider problem because the cost of inflation is shared by all countries. There debt constraints are desirable since they can restrain the temptation to inflate. The main difference between their paper and ours is that their time-inconsistency problem is rather a problem of coordination between different countries.

Athey, Atkenson and Kehoe (2005) consider a policymaker that can exploit private information about the state of the economy to create unexpected inflation and stimulate the economy. They find that the best incentive-compatible outcome can be implemented by legislating an upper limit on inflation. This is so because the optimal mechanism is static (the continuation values do not depend on type) and this upper limit restricts the policymaker in the states where it has a higher temptation to inflate. Our results suggest

that, in a more dynamic setting, where continuation values vary with type, an inflation cap could make cheating less painful. There the sustainability of the best outcome may require more constitutional restrictions than just an upper limit on inflation.

This paper also relates to the literatures on incomplete markets, private borrowing and sovereign default.¹⁰ Eaton and Gersovitz (1981) already point out that there is a natural borrowing limit beyond which it may not be feasible to repay the debt. The novelty of this paper is that it shows that this occurs in the public finance literature stemming from the Ramsey tradition and that a debt limit changes the characteristics of the worst sustainable equilibrium and, in turn, the set of equilibria that can be sustained.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 defines the set of competitive equilibria. Section 4 completes the description of the game and applies the APS method to our dynamic game. Section 5 studies the effect of institutional constraints on the level of government debt on the set of sustainable equilibria. Section 6 concludes. The appendix includes the description of the numerical method, proofs and figures.

2 The Environment

Our setup is very close to those of Lucas and Stokey (1983) and Chari and Kehoe (1993a,b). The economy is populated by a continuum (measure 1) of identical infinitely-lived individuals and by an infinitely-lived government. There is no capital accumulation. Output is a linear function of labor; per capita output is denoted y_t . The produced output is used for private and public consumption so that feasible allocations satisfy

$$c_t + g_t \leq y_t, \tag{1}$$

where c_t and g_t are the per capita levels of private and public consumption respectively.

The government is benevolent and its preferences coincide with those of the individuals, which take the following form:

$$\sum_{t=0}^{\infty} \beta^t [u(c_t, x_t) + z(g_t)], \tag{2}$$

where x_t is leisure and $\beta \in (0, 1)$ is the discount factor. The functions u and z are strictly increasing, strictly concave, continuously differentiable and satisfy the Inada conditions.

¹⁰See, for example, Reinhart et al. (2003) on sovereign debt intolerance.

Government consumption is financed through taxes on income and debt. In particular, we consider one-period bonds indexed to consumption.¹¹ More precisely, the government can offer a price q_t for a bond b_{t+1} to be bought by the individuals at date t , for which they receive a unit of consumption at date $t + 1$. This payment occurs unless the government decides to default on debt. The government's budget constraint is

$$g_t + (1 - \delta_t) b_t \leq \tau_t y_t + q_t b_{t+1}, \quad (3)$$

where $\tau_t \in [\underline{\tau}, \bar{\tau}]$ is the tax rate on income, with $\bar{\tau} < 1$, and $\delta_t \in \{0, 1\}$ is the default rate on government debt, which can take two values $\delta_t = 0$ (repayment) or $\delta_t = 1$ (default).¹² We also assume that the government has the ability to close down the market for government debt. This option is denoted by $\eta_t \in \{0, 1\}$, where $\eta_t = 1$ means that the market is open and $\eta_t = 0$ that the market is closed and no government bonds can be purchased.¹³ We denote the government policy at date t by $\pi_t = (g_t, \tau_t, \delta_t, \eta_t, q_t)$. We define Ω as the set of policies that satisfy the established institutional constraints. For the time being, this set is

$$\Omega \equiv \{ \pi \mid \tau \in [\underline{\tau}, \bar{\tau}], \text{ and } \delta \in \{0, 1\} \}.$$

Later Ω will also contain restrictions on the aggregate level of government debt and government deficit.

As commented earlier, the government can default on its debt obligations. We assume that such a default would bring about a direct cost to the economy, namely, a fall in productivity, as in Cole and Kehoe (2000). More specifically, the economy's technology - used by identical and perfectly competitive firms - is

$$y_t = \alpha_t (d - x_t),$$

where $d - x_t$ is the individual's labor supply ($d > 0$ is the time endowment per period).

¹¹Since the government does not have access to debt of all maturities, the time-inconsistency problem of manipulation of the interest rate cannot be solved via debt-restructuring. This assumption is not that severe in our setup. Domínguez (2005) considers a reputational mechanism and shows that, if there exist maturity structures that sustain the Ramsey outcome, such structures are generally simple with debt concentrated in few maturities.

¹²The upper bound on tax rates helps guarantee the equivalence of the recursive and sequence problem for the household. The lower bound is used in the numerical approximation of the value correspondence.

¹³This option is included for clarity. A government could always close down the market for debt by offering a price q_t for which no individuals would be willing to buy any bonds. Instead of that, we allow the government to choose whether to open the market for bonds or not and, if open, q_t clears the market.

The term α_t measures the productivity of the economy and satisfies

$$\alpha_t = \begin{cases} 1 - \omega_{t,s}, & \text{if the government has defaulted in period } s; \\ 1, & \text{if the government has never defaulted.} \end{cases}$$

We do not model the productivity cost $\omega_{t,s}$, but it could be an stationary autoregressive process so that the cost of a single default is only temporary.¹⁴ We assume $1 > \omega_{t,s} \geq 0$, with $\omega_{t,s} > 0$ for some $t \geq s$. Different explanations can be found to justify this cost. One could argue that a default on debt would generate a distrust in government policy that spills over the general productivity of the economy. Other arguments can be found in Cole and Kehoe (1998, 2000).

All households are identical, strategically anonymous, and behave identically along the equilibrium (we focus on symmetric equilibria). The household's budget constraint is

$$c_t + q_t b_{t+1} \leq (1 - \delta_t) b_t + (1 - \tau_t) \alpha_t (d - x_t), \quad (4)$$

where b_{t+1} satisfies $\underline{b} \leq b_{t+1} \leq \bar{b}$. The upper bound $\bar{b} > 0$ is assumed to be a very large finite number. The lower bound \underline{b} is set equal to zero. A justification for this is, as Chari and Kehoe (1993b), that claims against anonymous agents are not enforceable.¹⁵ The individual's allocation is a sequence $\{a_t\}_{t=0}^{\infty}$, where $a_t = (c_t, x_t, b_{t+1})$.

To complete the description of the environment we assume the existence of a public randomization device to ensure convexity. In particular, we consider an exogenous uncorrelated random variable Ψ_t that follows a uniform distribution in $[0, 1]$, whose realization ψ_t is publicly observed at the beginning of each period.

3 Competitive Equilibria

In this section we define the competitive equilibria for an arbitrary given policy. While government policies will not be set in this manner, the anonymity of each individual implies that their individual actions do not affect the actions of the government or other individuals and, in turn, they behave competitively. As in Chang (1998), Phelan and Stacchetti (2001) and Sleet (1997), we show that the set of competitive equilibria can

¹⁴We assume that the cost of default does not depend on the amount defaulted, as, for example, in Cole and Kehoe (2000) and Alesina, Prati and Tabellini (1990). Other papers, e.g. Calvo (1988), allow for that dependence. Later we discuss that possibility.

¹⁵Both bounds help ensure the equivalence of the recursive and sequence problem for the household.

be written recursively by incorporating as a new state variable the marginal value of the natural state variable, which, in our paper, is government debt.

We first present the sequence problem for the individual. Taking the policy and initial debt as given, he chooses consumption, leisure and bonds to maximize his welfare (2) subject to the budget constraint (4) and $0 \leq b_{t+1} \leq \bar{b}$. The first-order conditions are

$$u_x(c_t, x_t) = (1 - \tau_t) \alpha_t u_c(c_t, x_t), \quad (5)$$

$$u_c(c_t, x_t) q_t \geq \beta (1 - \delta_{t+1}) u_c(c_{t+1}, x_{t+1}), \text{ with equality if } b_{t+1} > 0, \quad (6)$$

where u_c denotes the partial derivative of u with respect to c ; other derivatives follow a similar notation. Note that, if default is expected, then $b_+ = 0$.

This sequence problem can be formulated recursively. To do so, we follow the methods suggested by Kydland and Prescott (1980) and Marcat and Marimón (1994). We first define the variable $m_s \equiv (1 - \delta_s) u_c(c_s, x_s)$, as the marginal value of government debt, and consider the following recursive problem:

$$\max_{(c_t, x_t, b_{t+1})} u(c_t, x_t) + z(g_t) + \beta m_{t+1} b_{t+1}, \text{ subject to (4) and } 0 \leq b_{t+1} \leq \bar{b}. \quad (\text{RP})$$

This problem yields the first-order condition (5) and $u_c(c_t, x_t) q_t \geq \beta m_{t+1}$, with equality if $b_{t+1} > 0$. The equivalence between the sequence and this recursive problem is guaranteed when: (i) $m_{t+1} \equiv (1 - \delta_{t+1}) u_c(c_{t+1}, x_{t+1})$; and (ii) the transversality condition

$$\lim_{t \rightarrow \infty} \beta^t m_t b_t = 0, \quad (7)$$

is satisfied. This is shown in the following proposition:

Proposition 1 *The recursive and the sequence problem for the individual are equivalent.*

Proof. See the Appendix ■

From now on we will denote k and k_+ as the current and future value of the variable k . We can summarize the competitive equilibrium conditions as follows:

Definition 1 (c, x, b_+, q) is a competitive equilibrium, $(c, x, b_+, q) \in CE(b, \tau, \delta, \eta, m_+)$, if and only if the following conditions are satisfied:

$$c + qb_+ = (1 - \delta) b + (1 - \tau) \alpha (d - x), \quad (\text{ce.1})$$

$$c + x + (\alpha \tau (d - x) + qb_+ - (1 - \delta) b) = d, \quad (\text{ce.2})$$

$$u_x(c, x) = (1 - \tau) \alpha u_c(c, x), \quad (\text{ce.3})$$

$$[u_c(c, x) q - \beta m_+] b_+ = 0, \quad [u_c(c, x) q - \beta m_+] \geq 0, \text{ and } b_+ \geq 0. \quad (\text{ce.4})$$

Some comments about the definition of competitive equilibrium are in order. First, this definition satisfies the government budget constraint, which has been implicitly introduced in (ce.2). Second, if $\eta_t = 0$, the market for government debt is closed and $b_+ = 0$. Finally, even though it is not explicit in the definition, the competitive equilibrium is a function of the history of random variables ψ_t .

4 The Game

Here we describe the dynamic game, denoted $\Gamma(b)$, between the government and the continuum of households. We assume that individuals are anonymous, i.e. individual decisions cannot be observed but only the aggregate outcomes. The non-observability of the individual actions implies that there is no strategic behavior from the part of the individuals. Moreover, this together with the convexity of the problem allow us to focus on symmetric strategy equilibria, where all individuals make the same choices along the equilibrium. For more details, see Phelan and Stacchetti (2001).

We denote the public history at the end of period $t - 1$ by $h_{t-1} = (\{\pi_s\}_{s=0}^{t-1}, \{\psi_s\}_{s=0}^{t-1})$, that is, the history of government policies and random outcomes. As discussed in Chari and Kehoe (1990), history does not need to include the distribution of individual actions. After ψ_t is realized, the government chooses first in any given date. A strategy for the government at date t , denoted $\sigma_{G,t}(h_{t-1}, \psi_t)$, is a choice of a current policy π_t as a function of the history h_{t-1} and the random outcome ψ_t . Then the individual makes its choice. A symmetric strategy for the individuals at date t , denoted $\sigma_{H,t}(h_t)$, is a choice of a current allocation a_t as a function of the public history $h_t = (h_{t-1}, \psi_t, \pi_t)$, which already includes the government's current policy. A symmetric strategy profile for $\Gamma(b)$ is a pair (σ_H, σ_G) , and the set of all $\sigma = (\sigma_H, \sigma_G)$ is denoted by $\Sigma = \Sigma_H \times \Sigma_G$. After each history h_{t-1} , a strategy profile $\sigma \in \Sigma$ induces a continuation strategy profile $\sigma|_{h_{t-1}}$.

The government's payoff is the value of $\sigma \in \Sigma$, given by

$$\Phi_G(b_0, \sigma) = (1 - \beta) E \left[\sum_{t=0}^{\infty} \beta^t [u(c_t, x_t) + z(g_t)] \right], \quad (8)$$

which is normalized by $(1 - \beta)$ to make it comparable to period utility. The household's payoff is the derivative at b of the value of σ , that takes the following value:

$$\Phi_H(b_0, \sigma) = (1 - \delta) u_c(c, x),$$

which represents the marginal value of government debt. Continuation payoffs are similarly defined for any history h_{t-1} with corresponding b_t and continuation strategy profile

$\sigma |_{h_{t-1}}$.

We define a sustainable equilibrium as follows:

Definition 2 *A symmetric strategy profile (σ_H, σ_G) of the game $\Gamma(b_0)$ is a sustainable equilibrium if it satisfies the following conditions for all $t \geq 0$: (i) given the strategy for the government $\sigma_{G,t}$, the continuation payoff for the household is higher than the payoff from any deviation to a different strategy $\sigma'_{H,t}$ for every history h_t ; and (ii) given the symmetric strategy for households $\sigma_{H,t}$, the continuation payoff for the government is higher than the payoff from any deviation to a different strategy $\sigma'_{G,t}$ for every history h_{t-1} .*

The definition of sustainable equilibrium is the same as the one in Chari and Kehoe (1990) and the one of symmetric sequential equilibrium in Phelan and Stacchetti (2001). This definition builds on two conditions that guarantee sequential rationality. The first condition implies that the government will not find profitable to deviate from its policy. The second condition says that individuals respond optimally to the government policy, which in turn implies a competitive equilibrium.

4.1 Self-Generation

Here we apply the APS method to our dynamic game. We will follow closely the work of Phelan and Stacchetti (2001) and we will only explain in detail the differences between their problem and ours. The main idea behind is that the set of sustainable equilibria can be written recursively by incorporating as new state variables: the promised continuation value for the government and the promised marginal value of debt for the household. Once we have a recursive problem, the set of values of all sustainable equilibria can be found as the largest fixed point of an operator.

In our model the state variable is government debt. We define an equilibrium value correspondence $V(b)$ that maps all possible $b \in [0, \bar{b}]$ into sets of payoffs for the household and government for which there exists a symmetric strategy profile which is a sustainable equilibrium. That is,

$$V(b_0) \equiv \{(\Phi_H(b_0, \sigma), \Phi_G(b_0, \sigma)) \mid \sigma \text{ is a sustainable equilibrium for } \Gamma(b_0)\}.$$

For a given b_0 , this correspondence is convex thanks to the presence of the random variable ψ_t . Given this randomization device, the strategies of government and individuals are functions of the history of ψ_t . For presentation purposes, we will not make explicit this dependence in our notation.

We also define an arbitrary value correspondence $W(b) : [\underline{b}, \bar{b}] \rightarrow R^2$ with compact and convex values. For a given value of b , the first dimension of this value correspondence is m , the marginal value of government debt. The second is v , the welfare or the government's payoff in a sustainable equilibrium. The boundaries of this correspondence are

$$\begin{aligned}\overline{W}(b, m) &\equiv \max_v \{v \mid (m, v) \in W(b)\}, \\ \underline{W}(b, m) &\equiv \min_v \{v \mid (m, v) \in W(b)\}.\end{aligned}$$

We now find a recursive formulation. For this, we need to introduce the concepts of consistency and admissibility with respect to the value correspondence.

Definition 3 A vector $\Lambda = (c, x, b_+, q, \tau, \delta, \eta, m_+, w_+)$ is consistent with respect to the value correspondence W at $b \in [\underline{b}, \bar{b}]$ if $(c, x, b_+, q) \in CE(b, \tau, \delta, \eta, m_+)$, $\pi \in \Omega$ and $(m_+, w_+) \in W(b_+)$.

For a given b , the vector Λ delivers a value

$$\Psi_G(b, \Lambda) \equiv (1 - \beta) [u(c, x) + z(\tau\alpha(d - x) + qb_+ - (1 - \delta)b)] + \beta w_+,$$

and a marginal value of debt

$$\Psi_H(b, \Lambda) \equiv (1 - \delta) u_c(c, x).$$

The functions Φ_i and Ψ_i , for $i = G, H$, are equivalent but defined over different domains. Let us also define the worst punishment as

$$\bar{\pi}_W(b) = \max_{\{\tau', \delta', \eta'\}} \min_{\{c', x', q', b'_+, m'_+, w'_+\}} \Psi_G(b, \Lambda') \text{ subject to} \quad (9)$$

Λ' is consistent with respect to W at b .

More explicitly, the worst punishment can be computed as

$$\begin{aligned}\bar{\pi}_W(b) = \max_{\{\tau', \delta', \eta'\}} \left\{ \min_{\{c', x', q', b'_+, m'_+, w'_+\}} (1 - \beta) [u(c', x') + z(\tau'\alpha(d - x') + q'b'_+ - (1 - \delta')b)] + \beta w'_+ \right\}, \\ \text{subject to } (c', x', b'_+, q') \in CE(b, \tau', \delta', \eta', m'_+), \\ \pi \in \Omega, \\ \text{and } (m'_+, w'_+) \in W(b'_+).\end{aligned}$$

If the government deviates and sets the policy τ' , δ' , η' , the punishment is the worst sustainable payoff. This punishment is sustainable because (i) the individuals respond optimally since Λ' is consistent with respect to W at b and, thus, we are in a competitive equilibrium; and (ii) the government sets the policy to maximize its payoff, therefore the government has no incentives to deviate. These two conditions are implicit in (9).

Since this is the worst sustainable payoff, any other sustainable equilibrium must provide a higher welfare. This is required in the definition of admissibility, which is the following:

Definition 4 *The vector Λ is admissible with respect to W at b if it is consistent with respect to W at b and*

$$\Psi_G(b, \Lambda) \geq \bar{\pi}_W(b). \quad (10)$$

For an arbitrary value correspondence $W(b)$, consistency and admissibility are separable. The equilibrium value correspondence will be shown to be a fixed point of an operator. This operator is defined as follows:

Definition 5 *For a value correspondence W , we define*

$$B(W)(b) = co(\{\Psi_G(b, \Lambda) \mid \Lambda \text{ is admissible with respect to } W \text{ at } b\}),$$

where $co(M)$ is for the convex hull of $M \subset R^2$.

Phelan and Stacchetti (2001) adapt Abreu, Pearce and Stacchetti (1990) procedure. First, they define:

Definition 6 *The set $W(b)$ is said to be self-generating if $W(b) \subseteq B(W)(b)$.*

Next they present the following results:

Theorem 1 *If W is self-generating, then $B(W) \subseteq V$.*

Proof. See Appendix. ■

Theorem 2 *If W is an upper semicontinuous (usc) value correspondence, then $B(W)$ is a usc value correspondence.*

Proof. See Appendix. ■

Lemma 1 *Graph(V) is a bounded set.*

Proof. See Appendix. ■

These theorems allow them to show their main result:

Theorem 3 *The equilibrium value correspondence V is the largest fixed point of B.*

Proof. See Appendix. ■

This allows them to extend the algorithm to compute the value correspondence:

Theorem 4 $W_\infty = V$.

Proof. See Appendix. ■

This algorithm is used in the next section to estimate the value correspondence.

5 Sustainable Equilibria and Institutional Restrictions on Government Debt

In this section we first characterize the set of sustainable equilibria for our basic economy. Then we investigate how institutional restrictions on government debt change the set of sustainable equilibria. In particular, we study the following two restrictions: limits on the level of government debt and limits on the budget deficit.

In our simple setup the worst sustainable equilibrium can be easily identified. Let Θ_{B1} and Θ_{B2} be partitions of the set of initial levels of debt such that $\Theta_{B1} = [\underline{b}, \tilde{B}]$, $\Theta_{B2} = [\tilde{B}, \bar{b}]$, $\tilde{B} \equiv \Theta_{B1} \cap \Theta_{B2}$ and $\Theta_{B1} \cup \Theta_{B2} = [\underline{b}, \bar{b}]$. We show that a balanced budget is the worst.

Proposition 2 *If $b \in \Theta_{B1}$, then the worst sustainable equilibrium is a balanced budget without default. Otherwise, if $b \in \Theta_{B2}$, the worst is a balanced budget with default.*

Proof. See the Appendix. ■

We find that, depending on the initial level of government debt, the worst sustainable equilibrium is either a balanced budget without default or a balanced budget with default. A balanced budget is clearly a sustainable equilibrium. Moreover, a balanced budget is the worst sustainable equilibrium because for any equilibrium to be sustainable the continuation payoff for the government must be higher than the payoff from any possible

deviation. If there exists a competitive equilibrium worse than a balanced budget, then the government can always deviate by imposing a balanced budget either through default or by closing down the market for bonds.

The two balanced budget equilibria of Proposition 2 have in common the inability to smooth the tax burden over time. However, they differ in the initial debt and the levels of productivity (due to default). As Chari and Kehoe (1993b) show, if there were no direct costs of default, then the worst would be a balanced budget with default for all initial debt positions. This is obvious because that equilibrium would have the same level of productivity but a lower level of debt than any other equilibria. However, for positive direct costs of default, region Θ_{B1} emerges, and the government prefers to impose a balanced budget without default.¹⁶

[Insert Figure 1 about here.]

The properties of the worst sustainable equilibrium shape the entire set of sustainable equilibria. This is illustrated in Figure 1, which shows the approximated value correspondence of our game.¹⁷ We approximate the value correspondence using a numerical implementation of the APS method. For $b = 0$, there is only one pair (m, v) that belongs to the value correspondence. This pair of values is that of the Ramsey, which is the best and the worst sustainable equilibrium. For positive but low levels of debt, the set of sustainable values contains more than one pair. The worst v decreases as debt increases and so does the best but not as rapidly. The lowest and highest sustainable m are increasing and strictly positive (in accordance with no default). For high levels of debt, the worst is characterized by default, the corresponding value v , denoted v^d , is low and constant and the corresponding m , denoted m^d , is obviously zero. As debt increases the set of values shrinks to the value of default and, for very high levels of debt, the pair (m^d, v^d) becomes the only sustainable equilibrium.

¹⁶We should discuss the ability of a government to impose a balanced budget without default. In our model this ability implies paying back all debt obligations right away and, for high levels of debt, that would not be an equilibrium. However, in a world with different debt maturities, it only implies to honor the debt obligations at the maturity period. Thus, in that context, the government has a much greater ability to impose a balanced budget without default.

¹⁷In all numerical examples we let $\theta \in (0, 1)$ and $\gamma > 0$ and assume the utility function

$$u(c_t, x_t) + z(g_t) = \theta \ln c_t + (1 - \theta) \ln x_t + \gamma \ln g. \quad (11)$$

5.1 A Limit on the Level of Government Debt

We now analyze how the set of sustainable equilibrium is affected by restrictions on the level of debt. We assume that the level of government debt cannot exceed some pre-agreed upper limit \bar{B} , that is,

$$b_+ \leq \bar{B}. \quad (12)$$

This limit \bar{B} , written in per capita terms, is an aggregate limit on government debt and does not restrict individual bond holdings. This changes our previous game by re-defining the set Ω so that

$$\Omega \equiv \{ \pi \mid \tau \in [\underline{\tau}, \bar{\tau}], \delta \in \{0, 1\} \text{ and } b_+ \leq \bar{B} \}.$$

Let us now define a level of debt \tilde{B} , such that $\tilde{B} \equiv \Theta_{B1} \cap \Theta_{B2}$ and let the debt limit $\bar{B} < \tilde{B}$. Then we find the following:

Proposition 3 *A debt limit $b_+ \leq \bar{B}$ changes the set of sustainable equilibria. More specifically,*

- (i) *default is no longer a sustainable equilibrium;*
- (ii) *the best sustainable equilibrium without the debt limit might not be sustainable.*

Proof. See the Appendix. ■

The imposition of a debt limit changes the set of sustainable equilibria. A debt limit allows the government to avoid some bad equilibria, such as default. In other words, the imposition of a debt limit makes default not credible. However, this affects the incentive compatibility constraint (10). Since the worst is now better, the best sustainable equilibrium without the debt limit may become not admissible. The best sustainable equilibrium with a debt limit may provide a lower welfare than the best sustainable equilibrium without the debt limit.

[Insert Figure 2 and Panel 1 about here.]

Figure 2 and Panel 1 illustrate this result. Figure 2 displays the approximated value correspondence with a debt limit. Now there is no default, all m are strictly positive and all v are strictly larger than v^d . This correspondence looks similar to the one of Figure 1. However, they are not the same. This is shown in Panel 1, where the value correspondence for $b = 6$ without (A) and with (B) a debt limit are presented. There we see that, without a debt limit (A.1), the best sustainable value is the Ramsey value. But, with a debt limit

(B.1), the Ramsey is not longer sustainable and the best provides a lower welfare. The continuation payoffs associated to the next period debt provide the explanation. Without the debt limit (A.2), the Ramsey allocation is characterized by higher debt. And for those levels of debt, the worst is default. The value of default v^d is so low that the Ramsey can be sustained. With the debt limit (B.2), the Ramsey plan (with a debt limit) implies an allocation with a lower level of debt. For this level of debt, the worst is not as bad. In fact, it provides such a high payoff that the incentive compatibility constraint (10) binds and the upper boundary of the value correspondence moves down. Now the Ramsey is not sustainable.

To sum up, a debt limit can solve the time-inconsistency problem of default. However, by doing so, the temptation to devalue government debt might make the government unable to sustain the best equilibrium. In the next section we explore whether restrictions on the government deficit can resolve the time-inconsistency problem of devaluation of government debt.

5.2 A Limit on the Government Budget Deficit

Our previous results show that a debt limit can credibly rule out default. In this section, we assume that there is no problem of default, $\delta_t = 0$, and examine whether restrictions on the budget deficit can help reduce the time-inconsistency problem of interest rate manipulation. As discussed by Lucas (1986), restrictions on the budget deficit may be effective devices to solve time-inconsistency problems. Therefore, we analyze the case where the government has a limit not only on the level of debt but on the budget deficit, that is,

$$\underline{Q} \leq qb_+ - b \leq \overline{Q}.$$

Under such a limit, the government might not be able to impose a balanced budget unilaterally. The set Ω is now defined as

$$\Omega \equiv \{ \pi \mid \tau \in [\underline{\tau}, \overline{\tau}], \delta = 0, b_+ \leq \overline{B} \text{ and } \underline{Q} \leq qb_+ - b \leq \overline{Q} \}.$$

Now the worst sustainable equilibrium cannot be identified for all levels of debt. In this case the worst sustainable payoff is computed using a numerical implementation of the APS method. Following this procedure we also obtain a numerical approximation of the equilibrium value correspondence.

[Insert Figures 4 and 5 about here.]

The effects of a restriction on the budget deficit are shown in Figures 4 and 5. Figure 4 depicts the approximation of the value correspondence with and without a restriction on the budget deficit. When there are no restrictions on the budget deficit, we have the situation described in the previous section. The worst sustainable value is higher than the default value but the Ramsey is not sustainable. The introduction of a loose restriction on the budget deficit changes the set of sustainable equilibria. Under this restriction, the government is not able to impose a balanced budget for high initial levels of debt. Since it cannot impose a balanced budget, the resulting worst sustainable equilibrium provides lower welfare. This punishment is now sufficiently severe to sustain the Ramsey value in equilibrium. Therefore, the imposition of a restriction on the budget deficit can eliminate some bad equilibria without throwing out some good ones.

The above result depends on the particular budget deficit constraint. Here we have assumed a loose constraint that allowed individuals to expect a quite bad outcome while impeded the government to impose a more convenient balanced budget for some levels of debt. The set of sustainable equilibria changes with the tightness of the balance budget constraint. As shown in Figure 5, tighter budget deficits reduce the set of competitive equilibria expected by the individuals, while they still impede the government to repay all debt right away. Therefore, we see that the tighter the budget the smaller the set of sustainable equilibria. Thus, a restriction on the budget deficit can be used to coordinate the beliefs of individuals and reduce the multiplicity of equilibria.

6 Conclusions

As Chari (1988) points out, it is the work of economists to design arrangements and constitutions that can help resolve the time-inconsistency problems of government policy. This paper assesses one type of institutional arrangement: debt limits, which have been recently adopted by many countries.

We have found that a limit on debt can be beneficial as it can eliminate some bad equilibria. In particular, if a government deviates from past announcements, it would do so through interest rate manipulation rather than default, and balanced budget with repayment, not default, characterizes the worst sustainable equilibrium. However, as the worst sustainable value is now higher, a limit on debt makes a deviation from the announced policy less painful and, in turn, may make the Ramsey outcome not sustainable.

We have also analyzed restrictions on the budget deficit to alleviate the time-inconsistency problem of interest rate manipulation. We have shown that the Ramsey can be sustained.

The main reason for this is that this additional restriction ties both the hands of the individuals and the government after a deviation and allows for more severe punishments than a balanced budget. The effects depend however on the specific budget deficit constraint. We find that the tighter the budget the smaller the set of sustainable equilibria. And, thus, a restriction on the budget deficit can be used to reduce the multiplicity of equilibria in this type of policy games.

Our results have implications for optimal taxation without commitment. Most of the work on this area either presumes that there is no debt or that default occurs after a deviation. As shown in Domínguez (2007), these assumptions are not innocuous and have an effect on the properties of optimal capital taxation. It is an open question how capital taxes should be set in this environment.

This paper emphasizes that the institutional environment transforms the reputational mechanism. Institutional constraints modify the set of strategies available and the public's expectations. Moreover, this effect depends on the economic features of the economy (as, for example, the direct costs of default). Therefore, these effects must be taken into account in the careful design of institutional restrictions. One question that this paper does not answer, however, is how the government can credibly commit to meet the terms of the institutional restriction.

References

- [1] Abreu, D., Pearce, D. and E. Stacchetti (1990). "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring." *Econometrica*, 58, 1041-1063.
- [2] Aiyagari, S. R., Marcet, A.; Sargent, T. and J. Seppälä (2002). "Optimal Taxation without State-Contingent Debt." *Journal of Political Economy*, 110, 1220-1254.
- [3] Alesina, A., Prati, A. and G. Tabellini (1990). "Public Confidence and Debt Management: A Model and a Case Study of Italy," in R. Dornbusch and M. Draghi (eds.), *Public Debt Management: Theory and History*, 94-124, Cambridge University Press.
- [4] Athey, S., Atkenson, A. and P. J. Kehoe (2005). "The Optimal Degree of Discretion in Monetary Policy." *Econometrica*, 73, 1431-1475.
- [5] Barro, R. J. and D. B. Gordon (1983). "Rules, Discretion and Reputation in a Model of Monetary Policy." *Journal of Monetary Economics*, 12, 101-121.

- [6] Buiter, W., Corsetti, G. and N. Roubini (1993). “Excessive Deficits: Sense and Nonsense in the Treaty of Maastricht.” *Economic Policy: A European Forum*, 8, 57-100.
- [7] Calvo, G. A. (1988). “Servicing the Public Debt: The Role of Expectations.” *American Economic Review*, 78, 647-661.
- [8] Chang, R. (1998). “Credible Monetary Policy in an Infinite Horizon Model: Recursive Approaches.” *Journal of Economic Theory*, 81, 431-61.
- [9] Chari, V. V. (1988). “Time Consistency and Optimal Policy Design.” *Quarterly Review of Federal Reserve Bank of Minneapolis*, 12, 4, 17-31.
- [10] Chari, V. V. and P. J. Kehoe (1990). “Sustainable Plans.” *Journal of Political Economy*, 98, 783-802.
- [11] Chari, V. V. and P. J. Kehoe (1993a). “Sustainable Plans and Debt.” *Journal of Economic Theory*, 61, 230-61.
- [12] Chari, V. V. and P. J. Kehoe (1993b). “Sustainable Plans and Mutual Default.” *Review of Economic Studies*, 60, 175-195.
- [13] Chari, V. V. and P. J. Kehoe (2007). “On the Desirability of Fiscal Constraints in a Monetary Union.” *Journal of Monetary Economics*, 54, 2399-2408.
- [14] Cole, H. L. and P. J. Kehoe (1998). “Models of Sovereign Debt: Practical vs. General Reputations.” *International Economic Review*, 29, 55-70.
- [15] Cole, H. L. and T. J. Kehoe (2000). “Self-Fulfilling Debt Crises.” *Review of Economic Studies*, 67, 91-116.
- [16] Cronshaw, M. (1997). “Algorithms for Finding Repeated Game Equilibria.” *Computational Economics*, 10, 139-168.
- [17] Domínguez, B. (2005). “Reputation in a Model with a Limited Debt Structure.” *Review of Economic Dynamics*, 8, 600–622.
- [18] Domínguez, B. (2007). “Public Debt and Optimal Taxes Without Commitment.” *Journal of Economic Theory*, 135, 159-170.
- [19] Eaton, J. and M. Gersovitz (1981). “Debt with Potential Reputation: Theoretical and Empirical Analysis.” *Review of Economic Studies*, 48, 289-309.

- [20] Fernández-Villaverde, J. and A. Tsyviski (2002). “Optimal Fiscal Policy in a Business Cycle Model without Commitment.” Working Paper.
- [21] Judd, K., Yeltekin S. and J. Conklin (2000). “Computing Supergame Equilibria.” *Econometrica*, 71, 1239-1255.
- [22] Krusell, P., Martín, F. M. and J. V. Ríos-Rull (2005). “Time-Consistent Debt.” Mimeo.
- [23] Kydland, F. E. and E. C. Prescott (1977). “Rules Rather than Discretion: The Inconsistency of Optimal Plans.” *Journal of Political Economy*, 85, 473-491.
- [24] Kydland, F. E. and E. C. Prescott (1980). “Dynamic Optimal Taxation, Rational Expectations and Optimal Control.” *Journal of Economic Dynamics and Control*, 2, 79-91.
- [25] Lucas, R. E. Jr. and N. L. Stokey (1983). “Optimal Fiscal and Monetary Policy in an Economy without Capital.” *Journal of Monetary Economics*, 12, 55-93.
- [26] Lucas, R. E. Jr. (1986). “Principles of Fiscal and Monetary Policy.” *Journal of Monetary Economics*, 17, 117-134.
- [27] Marcet, A. and R. Marimón (1994). “Recursive Contracts.” Mimeo.
- [28] Marcet, A. and A. Scott (2003). “Debt and Deficit Fluctuations and the Structure of Bond Markets.” *Journal of Economic Theory*, forthcoming.
- [29] Martín, F. M. (2006). “A Positive Theory of Government Debt.” Mimeo.
- [30] Phelan, C. and E. Stacchetti (2001). “Sequential Equilibria in a Ramsey Tax Model.” *Econometrica*, 69, 1491-1518.
- [31] Prescott, Edward C. (1977). “Should Control Theory be used for Economic Stabilization?” *Carnegie-Rochester Conference Series on Public Policy*. A supplementary series to the *Journal of Monetary Economics*. 13-38.
- [32] Reinhart, C. M., Rogoff, K. S. and M. A. Savastano (2003). “Debt Intolerance.” *Brookings Papers on Economic Activity*, 1, 1-74.
- [33] Rogoff, K. (1985). “The Optimal Degree of Commitment to an Intermediate Monetary Target.” *Quarterly Journal of Economics*, 100, 1169-1189.

- [34] Rogoff, K. (1987). "Reputational Constraints on Monetary Policy." *Carnegie-Rochester Conference Series on Public Policy*, 26, 141-182.
- [35] Sleet, C. (1997). "Recursive Methods for Solving Credible Government Policy Problems." Mimeo, KSM-MEDS, Northwestern University.
- [36] Sleet, C. and S. Yeltekin (2002). "On the Computation of Value Correspondence." Working paper, Hoover Institution.
- [37] Sleet, C. and S. Yeltekin (2006). "Optimal Taxation with Endogenously Incomplete Debt Markets." *Journal of Economic Theory*, 127, 36-73.
- [38] Stockman, D. (2004). "Default, Reputation and Balanced-Budget Rules." *Review of Economic Dynamics*, 7, 382-405.
- [39] Stokey, N. L. (1989). "Reputation and Time Consistency." *American Economic Review*, 79, 134-139.
- [40] Sturzenegger, F. (2004). "Tools for the Analysis of Debt Problems." *Journal of Reconstructing Finance*, 1, 1-23.
- [41] Towe, C. M. (1989). "Balanced Budget Constraints and the Time Consistency of Optimal Policy in a Non-Ricardian Economy." *Southern Economic Journal*, 55, 547-558.

7 Appendix: Proofs

Proof of Proposition 1

Here we prove that the transversality condition (7) is satisfied. In particular, we show that $m_t b_t$ is a uniformly bounded sequence, that is $m_t b_t \in (\underline{mb}, \overline{mb})$, where \bar{s} and \underline{s} respectively denote the maximum and minimum finite values of the variable s . By assumption, we have $b_t \in [\underline{b}, \bar{b}]$, then we just need to show that $m_t \in [\underline{m}, \overline{m}]$.

First, since $\delta \in \{0, 1\}$ and the marginal utility of consumption is strictly positive, we find a lower bound $\underline{m} = 0$.

Second, we show that $\overline{m} < \infty$. This is so because the individual can always guarantee strictly positive income. Consider the situation where the government has defaulted, imposes maximum tax rates $\bar{\tau} < 1$ and productivity is at its lowest possible level $\underline{\alpha} > 0$ (due to default). In this situation, given that $\lim_{c \rightarrow 0} u_c(c, x) = \infty$, the first order condition (5) implies that the household will optimally choose to work a strictly positive amount of time and earn a net income equal to $(1 - \bar{\tau})\underline{\alpha}(d - x) > 0$. Then, since income is strictly positive so is consumption $c > 0$ and, therefore, $\overline{m} < \infty$. Then we have $m_t b_t \in (0, \overline{mb})$. Therefore, $\lim_{t \rightarrow \infty} \beta^t m_t b_t = 0$. Hence, the recursive and the sequence problem are equivalent. ■

Proofs of Theorem 1-4 and Lemma 1

These proofs are a straightforward extension of those in Phelan and Stacchetti (2001) for our environment with government debt. The only difference comes at showing that $Graph(V)$ and $Graph(W)$ are bounded sets. We need to show that $graph(V) \subset [\underline{b}, \bar{b}] \times [\underline{m}, \overline{m}] \times [\underline{v}, \bar{v}]$ so that $graph(V)$ is bounded and similarly for $Graph(W)$. Since upper and lower bounds for m and b are given in the proof of Proposition 1, this amounts to prove that v is bounded.

Let's first work out a lower bound. Here we should be concerned about situations in which consumption of c , x , or g go to zero and the utility for the household might go to $-\infty$. In the proof of Proposition 1, we have shown that even if productivity is at its lowest level and taxes are maximum, the individual can always guarantee a minimum strictly positive level of income. Then the first order condition (5) together with the Inada conditions, imply that private consumption and leisure are bounded away from zero. Moreover, since not working, $x = d$, is never a solution, the government can always guarantee some strictly positive level of \underline{g} by choosing $\tau > 0$ and $\delta = 1$. Therefore, $\underline{v} = u(\underline{c}, \underline{x}) + z(\underline{g})$ is finite. Moreover, c , x , and g can never be greater than d , then an upper

bound for v is given by $\bar{v} = u(d, d) + z(d)$. Using the same arguments, we can construct similar upper and lower bounds for w_+ so that $graph(W) \subset [\underline{b}, \bar{b}] \times [\underline{m}, \bar{m}] \times [\underline{w}_+, \bar{w}_+]$.

■

Proof of Proposition 2

We first show that a balanced budget is a sustainable equilibrium. First, if the government chooses not to open the market for bonds, then the only decision of the individuals is the intraperiod consumption-leisure decision. This static equilibrium is obviously sustainable. Second, if the government's choice is to default $\delta = 0$, then it is in the best interest of the individuals not to buy any bonds. Moreover, if the direct cost of default is relatively low (compared to paying back the debt), a default reduces the need of distortionary taxation and would be optimal from the government's perspective. The worst sustainable equilibrium is the best of a balanced budget without default and with default. Suppose that there exists a sustainable equilibrium (σ', f') that provides less welfare than this balanced budget. Given that the government can always achieve a balanced budget unilaterally, the government would optimally deviate to the balanced budget. Thus, the pair (σ', f') fails to satisfy condition (ii) to be sustainable. ■

Proof of Proposition 3

From Proposition 2, we see that the properties of the worst sustainable equilibrium depend on the level of debt. Then, the imposition of a debt limit $\bar{B} < \tilde{B}$, where $\tilde{B} \equiv \Theta_{B1} \cap \Theta_{B2}$, restricts all possible levels of debt to the region Θ_{B1} . In this region, the worst sustainable equilibrium is a balanced budget without default. Thus, the government would never default and default is not longer a sustainable equilibrium.

Since the worst sustainable equilibrium yields a higher welfare, the incentive compatible constraint (10) is tighter and the best sustainable equilibrium without the debt limit may not be admissible (sustainable) with a debt limit. ■

8 Appendix: Numerical Approximation of the Value Correspondence

Here we explain the numerical approximation of the equilibrium value correspondence. The main references for this section include Cronshaw (1997), Judd, Yeltekin and Conklin (2000) and Sleet and Yeltekin (2001). Judd, Yeltekin and Conklin (2000) propose two alternative procedures for repeated games: an outer approximation and an inner approximation. We follow Phelan and Stacchetti (2001) and Fernández-Villaverde and Tsyviski (2002) to apply an outer approximation to our dynamic policy game.¹⁸

The outer approximation is as follows. First, we set an initial outer approximation $W^0(b_r)$ of $W(b_r)$ defined by an outward set of hyperplanes H and a vector of distances from the origin $C_{W(b_r)}^0$. A point $(m, w) \in W^0(b_r)$ if $H(m, w)' \leq C_{W(b_r)}^0$. Then, we compute the worst punishment $\bar{\pi}_W^1(b)$ given the guess of the value correspondence $W^0(b_r)$. We update the distances $C_{W(b_r)}^1$ and the value correspondence $W^1(b_r)$. This process is iterated until convergence. To apply this algorithm to our model, we follow the steps described in Fernández-Villaverde and Tsyviski (2002).

This algorithm consists of the following steps:

- **Step 0:** parameter values, discretization of the state space, fix $H \in R^{u,2}$ and initial distances $C_{W(b_r)}^0$.
 1. We choose the parameter values.
 2. We define a grid for debt $b = [b_1, b_2, \dots, b_{bn}]$ with $b_1 = 0$, $b_{bn} = \bar{B}$.
 3. To build the directional matrix H , we choose a u -point grid on $[0, 2\pi)$, $\frac{\theta}{1 \times u}$, which yields a matrix $H_{u \times 2}$.
 4. We fix an initial large distance $C_{W(b_r)}^0$ that satisfies $W^0(b_r) \subseteq V(b_r)$. We set the government's payoff under full-commitment as the maximum w , \bar{w} . Given the upper and lower bounds on taxes and public spending, it is easy to find minimum and maximum values for consumption and leisure. Using these values, we can fix some upper and lower bound for m and a minimum sustainable payoff w , \underline{w} .
- **Step 1:** we compute the worst punishment conditional on the current guess of the

¹⁸Fernández-Villaverde and Tsyviski (2002) also apply an inner approximation.

value correspondence. To do that, we solve

$$\bar{\pi}_W^1(b) = \max_{\{\tau', \delta', \eta'\}} \left\{ \min_{\{c', x', q', b'_+, m'_+, w'_+\}} (1 - \beta) [u(c', x') + z(\tau' \alpha(d - x') + q' b'_+ - (1 - \delta') b)] + \beta w'_+ \right\}$$

subject to $(c', x', b'_+, q') \in CE(b, \tau', \delta', \eta', m'_+)$,

$\pi \in \Omega$,

and $(m'_+, w'_+) \in W^0(b'_+)$.

Notice that now we have as a constraint that $(m'_+, w'_+) \in W^0(b'_+)$. Then, we can find a solution for the above problem.

- **Step 2:** update the distances. For each direction we find the maximum equilibrium values subject to competitive equilibrium conditions and sequential rationality. We update the distances solving for each direction h the program

$$C_{W(b_r)}^1(h) = \max_{c, x, b_+, q, \tau, \delta, \eta, m_+} H(h, 1) m + H(h, 2) w \text{ such that}$$

$$m = (1 - \delta) u_c(c, x),$$

$$(c, x, b_+, q) \in CE(b, \tau, \delta, \eta, m_+),$$

$$\pi \in \Omega,$$

$$w = (1 - \beta) [u(c, x) + z(\tau \alpha(d - x) + q b_+ - (1 - \delta) b)] + \beta w_+ \geq \bar{\pi}_W^1(b),$$

$$C_{W(b'_r)}^0(h) \geq H(m_+, w_+)$$

The last constraint means that the future payoffs must belong to the current guess of the equilibrium correspondence.

- **Step 3:** Update the guess of the equilibrium correspondence and iterate steps 1-2-3 until convergence.

9 Figures and Tables

For all figures and tables, we consider the utility function (11).

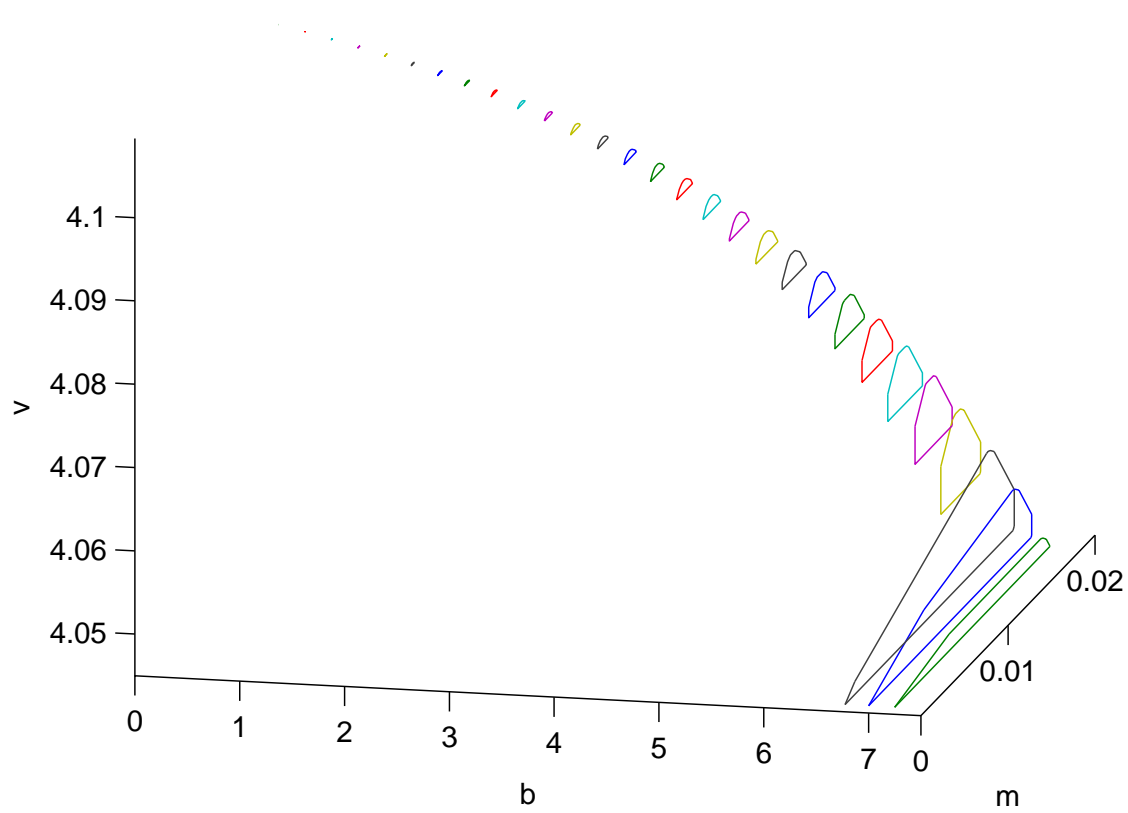


FIGURE 1. Equilibrium Value Correspondence.

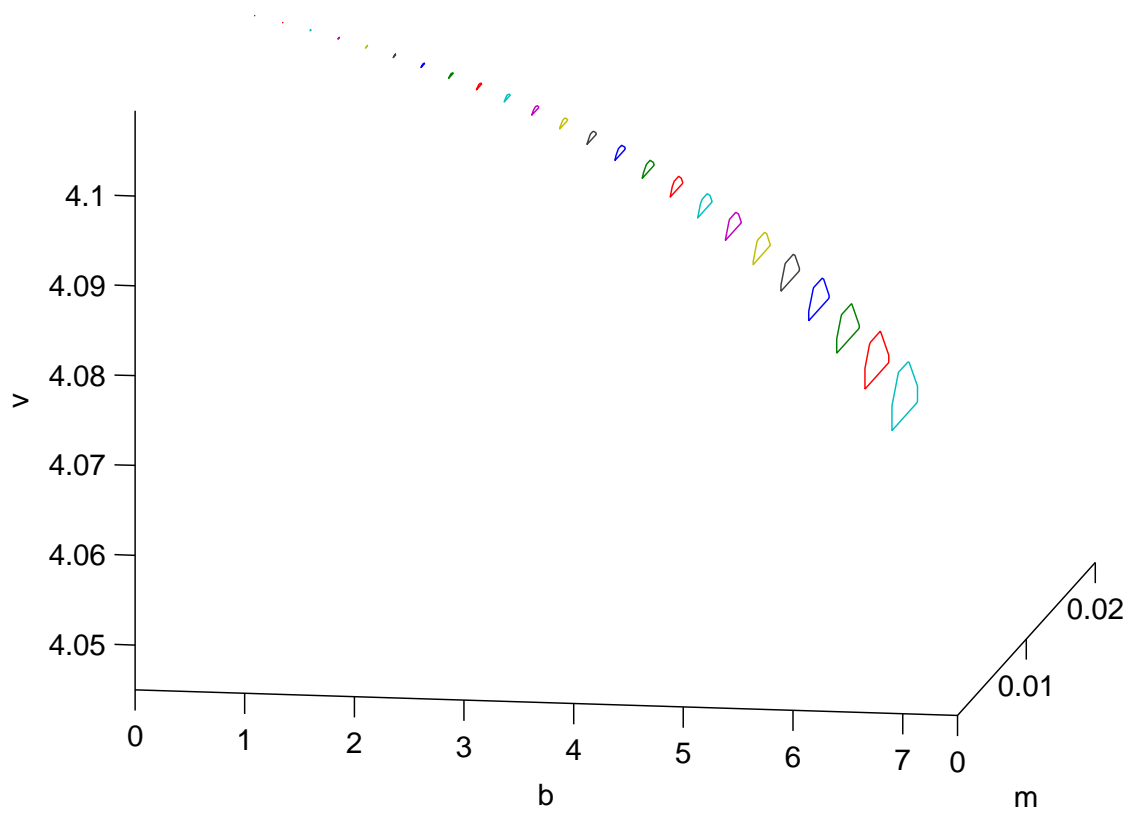
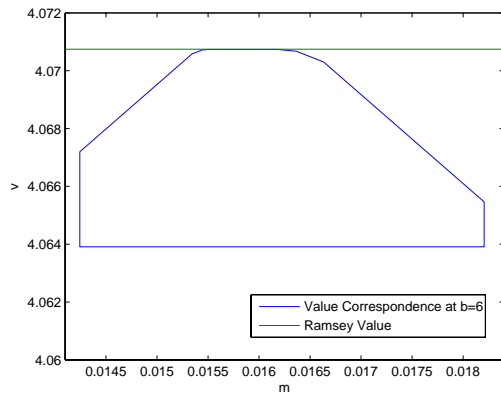


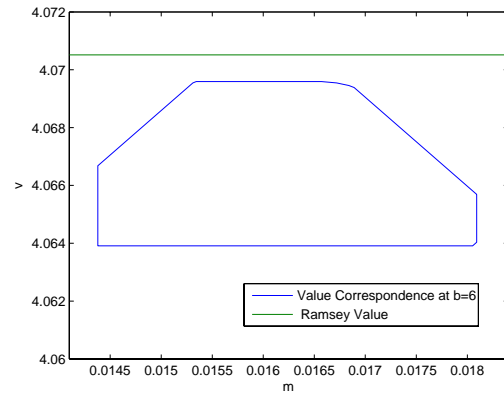
FIGURE 2. Equilibrium Value Correspondence with a Debt Limit at $\bar{B} = 6$.

PANEL 1. Value Correspondences at $b = 6$ without (A) and with (B) Debt Limit at $\bar{B} = 6$.

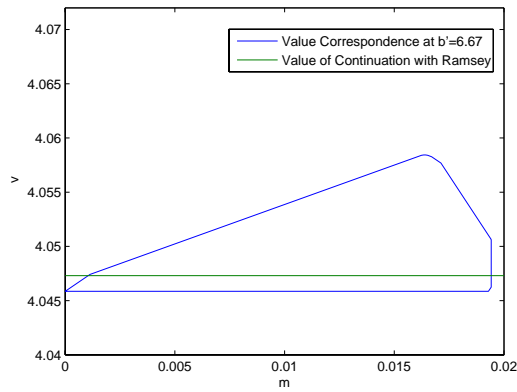
A. No Debt Limit



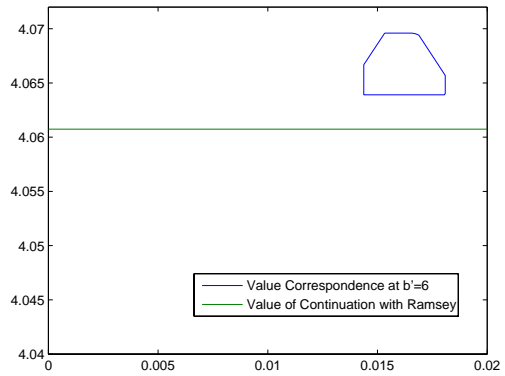
B. With Debt Limit



A1. Value Correspondence at $b = 6$ and Ramsey Value



B1. Value Correspondence at $b = 6$ and Ramsey Value



A2. Value Correspondence at $b_+ = 6.67$ and Value of Continuation with Ramsey

B2. Value Correspondence at $b_+ = 6$ and Value of Continuation with Ramsey

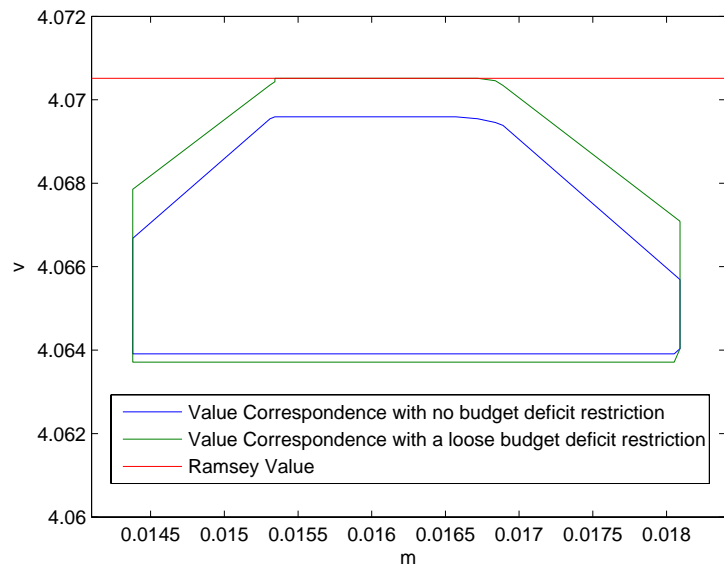


FIGURE 4. Set of Sustainable Values and Restrictions on the Budget Deficit

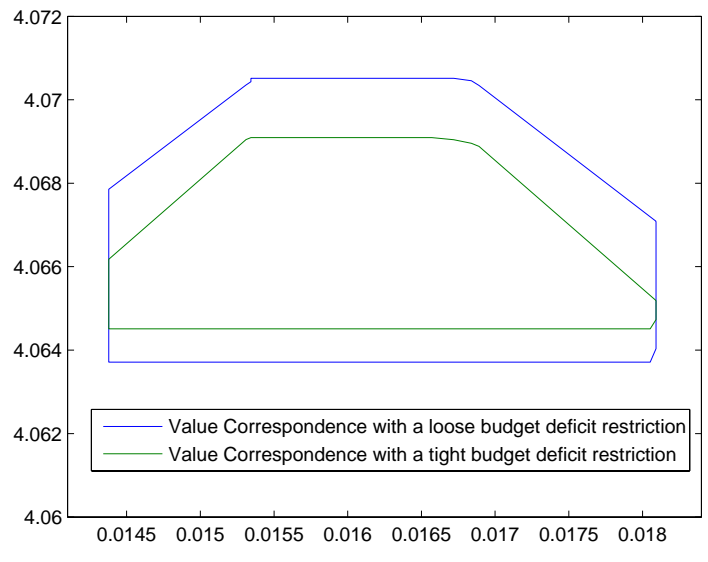


FIGURE 5. Set of Sustainable Values under Loose and Tight Restrictions on the Budget Deficit